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Additive ideal theory in multiplicative systems

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The theory of commutative or noncommutative multiplicative lattices (m-lattices or ideal lattices) has been developed by Krull [8], Birkhoff [3], Ward [16], Dilworth [5] [16], Curtis [4] and others. It is the purpose of the present paper to give some contributions to the theory of m-lattices and to that of r-ideals in multiplicative systems (m-systems).

In the first half of this paper we shall consider mainly a commutative but not necessarily associative m-lattice. As is well known, Ward-Dilworth [16] has defined a Noether lattice $\mathfrak{S}$ and obtained a primary decomposition theorem in $\mathfrak{S}$ as a lattice-formulation of the classical Lasker-Noether's theorem. Moreover they have discussed about sufficient conditions that a lattice is Noetherian. But the condition N_3 (every irreducible element in $\mathfrak{S}$ is primary) in [16] can be proved, if we make use of Birkhoff's somewhat modified residuals (Theorem 1). Hence we can propose that a commutative l-semigroup is Noetherian if it has a greatest unit and satisfies the ascending chain condition (a. c. c.). After a few preliminaries (§§1 and 2), we shall prove in §3 a primary decomposition theorem of elements in a non-associative m-lattice which is a generalization of the one of $\mathfrak{S}$. In §4 we consider the idempotents of a non-associative m-lattice (commutative or noncommutative). It is not true that the radical of a primary element is prime, but this is true if the multiplication is associative. In such a case isolated components of elements can be similarly discussed as in the case of rings, which is shown in §5. §6 shows a pairwise coprime decomposition of elements in a commutative m-lattice. The relation between the primary decomposition and the pairwise coprime decomposition is also exactly similar to the case of rings. In §7 we shall define r-ideals of a commutative but not necessarily associative m-system M as a generalization of Lorenzen's r-ideals of a commutative semigroup S [9]. By using the results in §§3 and 6, we obtain a short decomposition and a pairwise coprime decomposition of r-ideals in M (or in S). Schenkman [12] has recently pointed out the similarlity between the properties of ideals in a commutative ring and those of normal subgroups of a group, corresponding the product of ideals to the commutator-product of normal subgroups. From such a standpoint of view, we can consider naturally a decomposition of normal subgroups of a group on the analogy of the primary decomposition of ideals in a ring. But a close analogy breaks down, since it is not true that the radical of a primary normal subgroup is prime. We can obtain however a short representation theorem for normal subgroups of a group (§ 8). The proof of this theorem is carried out lattice-theoretically. The latter half of this paper deals with a primal decompositions of elements in a noncommutative l-semigroup K. We shall define an accessible subset of a lattice as a lattice-formulation of the set of all principal two-sided ideals in the lattice of all two-sided ideals of a ring with or without a unit. The accessible subset of K is fundamental when a. c. c. does not hold. In § 9 we define an NRP-element of an element in K, and define a weak NRP-element by using an accessible join generator system of K. In § 10 we shall give a definition of a primal element with its adjoint element, and prove that every element of K is represented as a finite or an infinite number of primal elements with maximal prime adjoints. The latter part of § 10 lays, under a. c. c. and the modularity of K, a short maximal (reduced) decomposition theorem by primal elements with maximal prime adjoints. If K is lower complete, then, by using the results in § 10, we can consider an isolated p-component of an element of K (§ 11). In the last section 12 we shall show, under some conditions, that the results in §§ 9, 10 and 11 are applicable to r-ideals in a semigroup. Taking r-operation as the module-generation of rings, we can see that these results are valid for two-sided ideals in a noncommutative ring with or without a unit element, and that they are analogous to the classical ideal theory in commutative rings. These results do not contain the Barnes' decomposition theorem [2], but they are a generalization of the Curtis' one [4].

§ 1. Ideals.

Throughout this paper we shall use L to denote a commutative multiplicative lattice (m-lattice)[1] with the following conditions:

C_1) L has the greatest element e.

C_2) L has the zero element 0.

C_3) $ab \leq a$ for any two elements $a, b \in L$.

We do not assume the multiplication to be associative, and the greatest element e to be (multiplicative) unit, except when we mention it particularly. If e is a unit, then the condition C_3) holds for L.

Let J be any (lattice-)ideal of L. Then J is a join-closed multiplicative ideal of L. An ideal generated by a subset S of L will be denoted by $j(S)$. Let J and J' be any two ideals of L. $J \vee J'$ will denote the ideal generated by J and J'. Evidently $J \vee J' = j(E)$, where $E = \{a \cup a'\,;\ a \in J, a' \in J'\}$. The intersection of J and J' will be denoted by $J \wedge J'$. It is easily verified that $J \wedge J' = \{a \cap a'\,;\ a \in J,$

1) Cf. [3; pp. 200–202].

$a' \in J'\}$. The multiplication $J \cdot J'$ of J and J' is defined by $j(M)$, where $M = \{aa'; a \in J, a' \in J'\}$.

The set $\mathfrak{L}$ of all ideals of L forms a commutative residuated lattice,[2] and L is imbeded in $\mathfrak{L}$ as an m-lattice. For, let J and J' be any two ideals of L. Then it is easily verified that the set $K = \{x; Jx \subseteq J', x \in L\}$ forms an ideal of L. Since $JK = \{ak; a \in J, k \in K\} \subseteq J'$, we have $J \cdot K = j(JK) \subseteq j(J') = J'$. If K' is an ideal such that $J \cdot K' \subseteq J'$, then evidently $K' \subseteq K$. Hence K is a residual of J' by J. Therefore $\mathfrak{L}$ forms a (commutative) residuated lattice under the set-inclusion relation and the multiplication $(\cdot)$. We now show that the set L^* of all principal ideals $j(a)$ in $\mathfrak{L}$ forms an m-sublattice of $\mathfrak{L}$. Evidently L^* forms a sublattice of $\mathfrak{L}$. We prove that $j(a) \cdot j(b) = j(ab)$. Take an arbitrary element x in $j(j(a)j(b))$. Then there exists a finite number of elements $u_1, \cdots, u_n \in j(a)j(b)$ such that $x \leq u_1 \cup \cdots \cup u_n$. Since $u_\nu \leq a_\nu b_\nu (a_\nu \in j(a), b_\nu \in j(b); \nu = 1, \cdots, n)$, we have $a_\nu \leq a$, $b_\nu \leq b$ and $x \leq ab \in j(ab)$. Hence $j(j(a)j(b)) \subseteq j(ab)$. The converse inclusion is evident. Hence $j(a) \cdot j(b) = j(j(a)j(b)) = j(ab)$. Therefore L^* is closed under multiplication, and L^* forms an m-sublattice of $\mathfrak{L}$. Since the mapping $a \to j(a)$ gives an isomorphism from L onto L^* as m-lattices, L^* can be identified with L.

The following properties are immediate:

(1) If any residual $j(a) : j(b)$ in $\mathfrak{L}$ is a principal ideal of L, then L forms a residuated lattice.

(2) If L is a cm-lattice, then it is a residuated lattice.

(3) If the ascending chain condition (a. c. c.) holds for L, then L is a residuated lattice.

Remark 1. Let $\mathfrak{L}^*$ be a residuated lattice consisting of all ideals of $\mathfrak{L}$, and $\mathfrak{J}$ be any element in $\mathfrak{L}^*$. Then the set $\mathfrak{J}^* = \{x; x \in J, J \in \mathfrak{J}\}$ forms an ideal of $L(\mathfrak{J}^* \in \mathfrak{L}^*)$. Conversely, for any element J of $\mathfrak{L}$, $J^\square = \sum_{A \subseteq J} j(A)$ is an element of $\mathfrak{L}^*$. It is easily verified that

$$J \to J^\square \to J^{\square *} = J, \quad \mathfrak{J} \to \mathfrak{J}^* \to \mathfrak{J}^{*\square} = \mathfrak{J}.$$

Hence $\mathfrak{L}^*$ can be identified with $\mathfrak{L}$.

§2. Powers.

Let a be any element of L. We define $a^{(1)} = a$ and $a^{(\rho)} = a^{(\rho-1)} \cdot a^{(\rho-1)}$ for any whole number $\rho > 1$. Some of the elementary properties of this power are listed here[3].

(1°) $a \leq b$ implies $a^{(\rho)} \leq b^{(\rho)}$.

(2°) $\rho \leq \sigma$ implies $a^{(\rho)} \geq a^{(\sigma)}$.

(3°) $a^{(\rho)(\sigma)} = a^{(\sigma)(\rho)}$.

2) Cf. [3; pp. 200-202].
3) Cf. [11].

(4°) $a^{(\rho+\sigma)}=a^{(\rho)(\sigma+1)}=a^{(\sigma)(\rho+1)}$.

(5°) $a^{(\rho\sigma)}\leq a^{(\rho)(\sigma)}$.

(6°) $(a\cap b)^{(\rho)}\leq a^{(\rho)}\cap b^{(\rho)}$.

(7°) $(a\cup b)^{(\rho\sigma)}\leq a^{(\rho)}\cup b^{(\sigma)}$.

If we define inductively $(ca)^{[1]}=ca$ and $(ca)^{[\rho]}=((ca)^{[\rho-1]})a$ for any whole number $\rho>1$, then we can prove

(8°) $(ca)^{(\rho)}\leq(ca)^{[\rho]}$.

DEFINITION 1. Let a be an element of L, and let $X_a=\{x;\ x^{(\rho)}\leq a,\ \rho=\rho(x),\ x\in L\}$. If there exists $\sup[X_a]$, then it is called a radical of a, and denoted by $\mathrm{rad}(a)$. If $\mathrm{rad}(a)=a$, a is called a radical element[4] of L.

LEMMA 1. *If a.c.c. holds for L, then there exists rad(a) for any element a of L, and* $(rad(a))^{(\lambda)}\leq a$ *for a suitable whole number* λ.

Proof. X_a forms an ideal of L. By a.c.c. X_a is principal: $X_a=j(a^*)$. Since $\mathrm{rad}(a)=a^*\in X_a$, we obtain $(\mathrm{rad}(a))^{(\lambda)}\leq a$ for a suitable λ.

LEMMA 2. *If a.c.c. holds for L, then rad(a) is a radical element for any element a of L.*

Proof. Since there exist whole numbers λ and ρ such that $(\mathrm{rad}(a))^{(\lambda)}\leq a$ and $(\mathrm{rad}(\mathrm{rad}(a)))^{(\rho)}\leq\mathrm{rad}(a)$, we have $(\mathrm{rad}(\mathrm{rad}(a)))^{(\rho\lambda)}\leq(\mathrm{rad}(\mathrm{rad}(a)))^{(\rho)(\lambda)}\leq(\mathrm{rad}(a))^{(\lambda)}\leq a$. Hence $\mathrm{rad}(\mathrm{rad}(a))\leq\mathrm{rad}(a)$. The converse inclusion is evident. We have therefore $\mathrm{rad}(\mathrm{rad}(a))=\mathrm{rad}(a)$.

§3. Primary decomposition.

An element p of L is called prime if $ab\leq p$ implies $a\leq p$ or $b\leq p$. An element q of L is called primary if $ab\leq q$ and $a\not\leq q$ imply $b^{(\sigma)}\leq q$ for a suitable whole number σ.

In this section we shall assume that a.c.c. holds for L. Then by (3) in §1, L forms a residuated lattice.

LEMMA 3. *Let a be any element of L. Then any prime element containing a contains rad(a).*

Proof. Let p be any prime element containing $\mathrm{rad}(a)$. Since $(\mathrm{rad}(a))^{(\lambda)}\leq a\leq p$ for some λ, we have $\mathrm{rad}(a)\leq p$, Q.E.D.

Let q^* be a radical of a primary element q of L. It is not true that q^* is a prime element[5]. But we shall say for convenience that q is q^*-primary, or that q belongs to q^*. Suppose that q belongs to q^*. Then the following properties are immediate:

(1) If $ab\leq q$ and $a\not\leq q^*$, then $b\leq q$.

(2) If $a\not\leq q^*$, then $q:a=q$.

4) Cf. [11].

5) Cf. [14; p. 379].

LEMMA 4. *Suppose that u and u^* are two elements of L, for which the following three conditions are satisfied*:

(α) $u \leq u^*$.

(β) $u^{*(\rho)} \leq u$ *for a suitable whole number* ρ.

(γ) *If* $ab \leq u$ *and* $a \not\leq u$, *then* $b \leq u^*$.

Then u is a u^*-primary element of L.

Proof. Let a and b be two elements such that $ab \leq u$ and $a \not\leq u$. Then by (γ) and (β), $b^{(\rho)} \leq u$; that is, u is a primary element. Put $u' = \mathrm{rad}(u)$. Then by (β), $u^* \leq u'$. Let λ be a whole number such that $u'^{(\lambda)} \leq u$. Then $u'^{(\lambda)} \leq u^*$. Suppose that λ is minimal. If $\lambda \neq 1$, then of course $u'^{(\lambda-1)} \not\leq u$ and $u'^{(\lambda-1)} \cdot u'^{(\lambda-1)} = u'^{(\lambda)} \leq u$, This implies $u'^{(\lambda-1)} \leq u^*$, and we have a contradiction. Hence $\lambda = 1$, i.e., $u' \leq u^*$. We get therefore $u' = u^*$, and u is u^*-primary.

LEMMA 5. *Let q be a q^*-primary element of L. Then the set of all q^*-primary elements of L forms a meet-semilattice.*

Proof. Let q_1 and q_2 be any two q^*-primary elements of L. Then it is easy to see that $q_1 \frown q_2$ and q^* satisfy the conditions (α), (β) and (γ) in Lemma 4.

THEOREM 1. *Any meet-irreducible element of L is a primary element.*

Proof. Suppose that an element a is not primary. Then there exist two elements u and v such that $uv \leq a$, $u \not\leq a$ and $v^{(\sigma)} \not\leq a$ for every whole number σ. For convenience, we shall use a symbol $(a : v)^{\{\nu\}}$ for the ν-tuple residuation:

$$(\cdots((a : \underbrace{v) : v) : \cdots) : v}_{\nu\text{-tuple.}}$$

Make an ascending chain

$$a : v \leq (a : v)^{\{2\}} \leq (a : v)^{\{3\}} \leq \cdots .$$

Then by a.c.c. for L, there exists a whole number λ such that $(a : v)^{\{\lambda\}} = (a : v)^{\{\lambda+1\}} = \cdots$. Put $W = \{(zv)^{[\lambda]}; z \in L\}$ and $K = \{x \smile w; x \in j(a), w \in W\}$. We now prove that $j(a)$ is equal to the intersection $j(a \smile u) \wedge K$[6]. Since it is easily verified that $j(a)$ is contained in K, we have $j(a) \subseteq j(a \smile u) \wedge K$. Take an arbitrary element c in $j(a \smile u) \wedge K$. Then $c \leq a \smile u$ and $c = x \smile (yv)^{[\lambda]}$, where $x \in j(a)$ and $(yv)^{[\lambda]} \in W$. Hence $(yv)^{[\lambda+1]} \smile xv = ((yv)^{[\lambda]} \smile x)v = cv \leq (a \smile u)v = av \smile uv \leq a \smile a = a$, i.e., $(yv)^{[\lambda+1]} \leq a$. Hence $y \leq (a : v)^{\{\lambda+1\}} = (a : v)^{\{\lambda\}}$, hence $(yv)^{[\lambda]} \leq a$, and hence $c = x \smile (yv)^{[\lambda]} \leq a$. We get therefore $j(a \smile u) \wedge K \subseteq j(a)$, $j(a \smile u) \wedge K = j(a)$. It is easy to see that the element $(a \smile u) \frown (a \smile (ev)^{[\lambda]})$ is the greatest element of $j(a \smile u) \wedge K$. Hence $a = (a \smile u) \frown (a \smile (ev)^{[\lambda]})$. Since $a < a \smile v^{(2\lambda)} \leq a \smile v^{(2)(\lambda)} = a \smile (vv)^{(\lambda)} \leq a \smile (ev)^{(\lambda)} \leq a \smile (ev)^{[\lambda]}$, a is strictly contained in $a \smile (ev)^{[\lambda]}$. It is evident that a is also strictly contained in $a \smile u$. Therefore a is meet-reducible. This completes the proof.

6) K is not an ideal of L, but since K contains the zero, $j(a \smile u) \wedge K$ forms an ideal.

THEOREM 2. *Any element a of L is decomposed as a meet of a finite number of primary elements q_i, i.e., $a=q_1 \cap \cdots \cap q_n$, and $rad(a)=rad(q_1) \cap \cdots \cap rad(q_n)$.*

Proof. The first part of the theorem is easily obtained by a.c.c. and by Theorem 1. We now prove the later part. Since there exists a whole number λ such that $(\mathrm{rad}(a))^{(\lambda)} \leq a \leq q_i$, we have $\mathrm{rad}(a) \leq \mathrm{rad}(q_i)$, $i=1, \cdots, n$. Hence $\mathrm{rad}(a) \leq \mathrm{rad}(q_1) \cap \cdots \cap \mathrm{rad}(q_n)$. On the other hand, since $c \equiv \mathrm{rad}(q_1) \cap \cdots \cap \mathrm{rad}(q_n) \leq \mathrm{rad}(q_i)$, i.e., $c^{(\rho_i)} \leq q_i$ for suitable whole numbers ρ_i, $i=1, \cdots, n$, we have $c^{(\rho)} \leq q_1 \cap \cdots \cap q_n = a$, where $\rho = \mathrm{Max}\{\rho_1, \cdots, \rho_n\}$. Hence $c \leq \mathrm{rad}(a)$. We have therefore $\mathrm{rad}(a) = \mathrm{rad}(q_1) \cap \cdots \cap \mathrm{rad}(q_n)$, as desired.

REMARK 2. Any element p containing $a=q_1 \cap \cdots \cap q_n$ contains at least one of the $\mathrm{rad}(q_i)$. For, since $(\cdots((q_1q_2)q_3)\cdots)q_n \leq q_1 \cap \cdots \cap q_n = a \leq p$, there exists q_i such that $q_i \leq p$. Hence $\mathrm{rad}(q_i) \leq \mathrm{rad}(p) = p$.

By Theorem 2 and Lemma 5, any element a of L is decomposed as a meet of a finite number of primary elements $q_i (i=1, \cdots, m)$ with the following conditions:

1°) $a=q_1 \cap \cdots \cap q_m$ is irredundant.

2°) $\mathrm{rad}(q_1), \cdots, \mathrm{rad}(q_m)$ are all different.

Such a decomposition is called a short or normal decomposition of a.

THEOREM 3. *Suppose that $a=q_1 \cap \cdots \cap q_m = q'_1 \cap \cdots \cap q'_n$ are two short representations of a. Then $m=n$, and it is possible to number the components in such a way that $rad(q_i)=rad(q'_i)$ for $i=1, \cdots, m=n$.*

Proof. Take a maximal radical in the set $\{\mathrm{rad}(q_1), \cdots, \mathrm{rad}(q_m), \mathrm{rad}(q'_1), \cdots, \mathrm{rad}(q'_n)\}$. We may suppose, without loss of generality, that the maximal radical is $\mathrm{rad}(q_1)$. Now we prove that $\mathrm{rad}(q_1)$ occurs among $\mathrm{rad}(q'_k)$, $k=1, \cdots, n$. Assume that $\mathrm{rad}(q_1) \neq \mathrm{rad}(q'_k)$ for all $k=1, \cdots, n$. Then, using the assumption, we can prove that $q_1 \not\leq \mathrm{rad}(q'_k)$ for $k=1, \cdots, n$. For, if $q_1 \leq \mathrm{rad}(q'_k)$ for some k, then $\mathrm{rad}(q_1) \leq \mathrm{rad}(\mathrm{rad}(q'_k)) = \mathrm{rad}(q'_k)$. Since $\mathrm{rad}(q_1) \neq \mathrm{rad}(q'_k)$, $\mathrm{rad}(q_1) < \mathrm{rad}(q'_k)$. This is a contradiction. It is easily verified that $q_1 \not\leq \mathrm{rad}(q_i)$ for $i=2, \cdots, m$. Hence $q_i : q_1 = q_i$ for $i=2, \cdots, m$, and $q'_k : q_1 = q'_k$ for $k=1, \cdots, n$. Hence by Theorem 3-(i) in [3] $a=q'_1 \cap \cdots \cap q'_n = (q'_1 : q_1) \cap \cdots \cap (q'_n : q_1) = (q_1 : q_1) \cap (q_2 : q_1) \cap \cdots \cap (q_m : q_1) = e \cap q_2 \cap \cdots \cap q_m = q_2 \cap \cdots \cap q_m$, and this is a contradiction.

We can now suppose, without loss of generality, $\mathrm{rad}(q_1) = \mathrm{rad}(q'_1)$, and make

$$(*) \qquad (q_2 : q_1) \cap \cdots \cap (q_m : q_1) = (q'_1 : q_1) \cap \cdots \cap (q'_n : q_1) .$$

It is easily verified, by the maximality of $\mathrm{rad}(q_1) = \mathrm{rad}(q'_1)$, that $q_1 \not\leq \mathrm{rad}(q_\nu)$ for $\nu \neq 1$, and $q_1 \leq \mathrm{rad}(q'_\mu)$ for $\mu \neq 1$. Hence we have $q_\nu : q_1 = q_\nu$ $(\nu \neq 1)$, and $q'_\mu : q_1 = q'_\mu$ $(\mu \neq 1)$. Hence by $(*)$ we have $q_2 \cap \cdots \cap q_m = (q'_1 : q_1) \cap q'_2 \cap \cdots \cap q'_n$, and have

$$(**) \qquad (q_2 : q'_1) \cap \cdots \cap (q_m : q'_1) = ((q'_1 : q_1) : q'_1) \cap (q'_2 : q'_1) \cap \cdots \cap (q'_n : q'_1) .$$

Since it is easily verified that $q'_1 \leq \mathrm{rad}(q_\nu)$ for $\nu=2, \cdots, m$, and $q'_1 \not\leq \mathrm{rad}(q')$ for

$\mu=2,\cdots,n$, and since $q_1':q_1\geq q_1'$, we have $q_\nu:q_1'=q_\nu$ $(\nu\neq 1)$, $q_\mu':q_1'=q_\mu'$ $(\mu\neq 1)$ and $(q_1':q_1):q_1'=e$. Hence by (**), we have

$$(***)\qquad q_2\cap\cdots\cap q_m=q_1'\cap\cdots\cap q_n'.$$

Continuing an exactly similar argument for (***), we obtain after a finite number of steps that $m=n$, and $\mathrm{rad}(q_i)=\mathrm{rad}(q_i')$ for $i=1,\cdots,m=n$.

§4 Idempotents.

In this section we shall assume that a. c. c. holds for L.

An element a of L is called an idempotent, if $a^{(2)}=a\cdot a=a$. If a is an idempotemt, then $a^{(\sigma)}=a$ for every whole number σ. If $a^{(\sigma)}=a$ for some whole number $\sigma>1$, then a is an idempotent of L. Let a and b be two idempotents of L. Then there exists a whole number ρ such that $a\cap b^{(\rho)}\leq ab$. For, let $ab=q_1\cap\cdots\cap q_n$ be a decomposition of ab into primary elements q_i. Then $ab\leq q_i$ for $i=1,\cdots,n$. Hence we can suppose, without loss of generality, that $a\leq q_i$ for $i=1,\cdots,r$, and $a\not\leq q_k$ for $k=r+1,\cdots,n$ $(0\leq r\leq n)$. Take whole numbers ρ_k such that $b^{(\rho_k)}\leq q_k(k=r+1,\cdots,n)$, and put $\rho=\mathrm{Max}\{\rho_{r+1},\cdots,\rho_n\}$. Then, since $b^{(\rho)}\leq q_{r+1}\cap\cdots\cap q_n$, we have $a\cap b^{(\rho)}\leq(q_1\cap\cdots\cap q_r)\cap(q_{r+1}\cap\cdots\cap q_n)=ab$.

Let c be an idempotent, x, y be two arbitrary elements of L. Then by the above arguments we have

(1) $cx=c\cap x$.

(2) $c(x\cap y)=cx\cap cy$.

(3) Any idempotent of L is neutral.

THEOREM 4. *The set of all idempotents of any commutative m-lattice L with a. c. c. forms a distributive lattice. Any idempotent of L is uniquely decomposed as a meet of a finite number of idempotents which are not decomposed as a meet of a finite number of idempotents except a trivial case.*

Proof. This is immediate.

From now on $\mathfrak{M}$ will denote an m-lattice with the following conditions:

1. $\mathfrak{M}$ has the greatest element e as an idempotent.
2. $ab\leq a$ and $ab\leq b$ for any $a,b\in\mathfrak{M}$.

The multiplication of $\mathfrak{M}$ is not necessarily associative and commutative.

For any element a of $\mathfrak{M}$, we shall define inductively $a^{(1)}=a$, $a^{(\rho)}=a^{(\rho-1)}a^{(\rho-1)}$ for $\rho>1$. Then we can prove the properties $(1^\circ),\cdots,(7^\circ)$ obtained in §2.

Let Δ_a be the set of all x which satisfy $a^{(\rho)}\leq x\leq a$ for a suitable whole number ρ.

LEMMA. 6. *Δ_a forms an m-sublattice of $\mathfrak{M}$.*

Proof. If $a^{(\rho)}\leq x\leq a$ and $a^{(\sigma)}\leq y\leq a$, then $a^{(\rho\sigma)}=(a\cup a)^{(\rho\sigma)}\leq a^{(\rho)}\cap a^{(\sigma)}\leq x\cup y\leq a$, $a^{(2\rho\sigma)}\leq a^{(2)(\rho\sigma)}=(aa)^{(\rho\sigma)}=a^{(\rho\sigma)}a^{(\rho\sigma)}\leq a^{(\rho)}a^{(\sigma)}\leq xy\leq x\cap y\leq a$.

LEMMA 7. *Let a be an idempotent of $\mathfrak{M}$. If $a=b\cap c$, then $a=b'\cap c'=b'c'=c'b'$ for any $b'\in\Delta_b$ and any $c'\in\Delta_c$. In particular $a=bc=cb$.*

Proof. Since $b^{(\rho)} \leq b' \leq b$ and $c^{(\sigma)} \leq c' \leq c$ for suitable whole numbers ρ and σ, we have $a = a^{(\rho\sigma)} = (aa)^{(\rho\sigma)} = a^{(\rho\sigma)}a^{(\rho\sigma)} \leq a^{(\rho)}a^{(\sigma)} \leq b^{(\rho)}c^{(\sigma)} \leq b'c' \leq b' \frown c' \leq b \frown c = a$. Hence $a = b'c' = b' \frown c' = c'b'$.

LEMMA 8. *Let a be an idempotent of $\mathfrak{M}$. If $a = a_1 \frown \cdots \frown a_\lambda$, then a is equal to any product of $c_1, \cdots, c_\lambda$, where c_i is an arbitrary element of $\Delta_{a_i}(i=1, \cdots, \lambda)$. In particular a is equal to any product of a_i $(i=1, \cdots, \lambda)$.*

Proof. By induction on λ, we can prove easily.

If $a = a_1 \frown \cdots \frown a_\lambda$, then by Lemma 8, we can write $a = a_1 \cdots a_\lambda$.

LEMMA 9. *Let a be an idempotent of $\mathfrak{M}$. If $a \leq c$, then $\Delta_c \subseteq [e, a]$.*

Proof. If $c^{(\rho)} \leq x \leq c$ for a whole number ρ, then $a = a^{(\rho)} \leq c^{(\rho)} \leq x$, i.e., $x \in [e, a]$.

LEMMA 10. *Let a be an idempotent of $\mathfrak{M}$, and suppose that the descending chain condition*[7] *holds for $[e, a]$. If $a \leq u$, then $u_0 = \inf[\Delta_u]$ is an idempotent, and contained in Δ_u.*

Proof. Let x be any element of Δ_u, and $\rho^* = \rho^*(x)$ be the smallest whole number such that $u^{(\rho^*)} \leq x \leq u$. We now prove that the set N_u consisting of all $\rho^*(x)$ $(x \in \Delta_u)$ is bounded. Suppose that N_u is not bounded. Then we can take infinite countable elements $x_1, x_2, \cdots$ of Δ_u such that

$$\rho_1^* < \rho_2^* < \cdots < \rho_n^* < \cdots,$$

where $\rho_n^* = \rho^*(x_n)$. Then $u^{(\rho_1^*)} > u^{(\rho_2^*)} > \cdots$. Because, if $u^{(\rho_n^*)} = u^{(\rho_{n+1}^*)}$, then $u^{(\rho_n^*)} \leq x_{n+1} \leq u$, and this is a contradiction. Evidently $\{u^{(\rho_n^*)}; n = 1, 2, \cdots\}$ is contained in Δ_u, and Δ_u is contained in $[e, a]$. This contradicts the descending chain condition for $[e, a]$.

Take an upper bound ρ_0 of N_u. Then $u^{(\rho_0)} \leq x \leq u$ for every element x of Δ_u. Hence $u^{(\rho_0)} \leq \inf[\Delta_u] \leq u$, i.e., $u_0 = \inf[\Delta_u] \in \Delta_u$. Since $u_0^2 \in \Delta_u$ and $u_0^2 \leq u_0$, we have $u_0^2 = u_0$, completing the proof.

THEOREM 5. *Let a be an idempotent of $\mathfrak{M}$, and suppose that the closed interval $[e, a]$ has finite length. Then a can be decomposed as a meet of a finite number of meet-irreducible idempotents $a_1, \cdots, a_n$, and a is equal to an arbitrary product of $a_1, \cdots, a_n$.*

Proof. If a is meet-irreducible, then there is nothing to prove. If a is meet-reducible, the theorem follows readily by Lemmas 8 and 10.

LEMMA 11. *Let f be an idempotent of $\mathfrak{M}$. If $f = b \smile c$, then $f = b' \smile c'$ for any $b' \in \Delta_b$ and any $c' \in \Delta_c$.*

Proof. Since $b^{(\rho)} \leq b' \leq b$ and $c^{(\sigma)} \leq c' \leq c$ for suitable whole numbers ρ and σ, we have $f = f^{(\rho\sigma)} = (b \smile c)^{(\rho\sigma)} \leq b^{(\rho)} \smile c^{(\sigma)} \leq b' \smile c' \leq b \smile c = f$. Hence we obtain $f = b' \smile c'$.

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7) If $\mathfrak{M}$ is associative, then this is equivalent to a. c. c. under some conditions. Cf. [10].

THEOREM 6. *If a is an idempotent such that the closed interval $[e, a]$ has finite length, then a is represented as a meet (or product) of finite number of pairwise coprime idempotents, none of which has such representation.*

Proof. If a is not represented as a meet of a finite number of pairwise coprime elements, then there is nothing to prove. Otherwise, using Lemmas 7, 10 and 11, we obtain $a=b_0\cap c_0$, $b_0\cup c_0=e$, $b_0^2=b_0\neq e$, $c_0^2=c_0\neq e$. By an induction argument on the length of $[e, a]$, we can assume that b_0 and c_0 have decompositions of pairwise coprime idempotents, hence a has such a decomposition.

§5. Isolated components.

In this section we shall assume that the multiplication of L is associative. Then evidently $a^{(\rho)}=a^{2^{\rho}-1}$.

LEMMA 12. *Let q be a primary element of L. Then $p=rad(q)$ is a prime element of L.*

Proof. Suppose that $ab\leq p$ and $a\nleq p$. Then $a^{\rho}b^{\rho}=(ab)^{\rho}\leq q$ for a suitable whole number ρ. We have then $b^{\sigma}\leq q$ for some σ, i.e., $b\leq \mathrm{rad}(q)=p$, Q.E.D.

DEFINITION 2. A prime element p is called a *minimal prime of a*, if (1) $p\geq a$ and (2) there is no prime p' such that $p>p'>a$.

We now assume that a.c.c. holds for L.

THEOREM 7. *Let $a=q_1\cap\cdots\cap q_n$ be a short representation of a. Then any minimal prime of a coincides with some $rad(q_i)$. Rad (a) is the meet of all minimal primes of a.*

Proof. The former part is immediate by Remark 2 in §3 and Lemma 12. The later part is easy to see.

THEOREM 8. *Let $a=q_1\cap\cdots\cap q_m$ and $b=q_1'\cap\cdots\cap q_n'$ be short representations of a and b. Then the following conditions are equivalent:*

1) $a:b=a$.
2) $\mathrm{rad}(q_i)\ngeq b$ *for* $i=1,\cdots,m$.
3) $\mathrm{rad}(q_i)\ngeq \mathrm{rad}(q_k')$ *for* $i=1,\cdots,m$; $k=1,\cdots,n$.

Proof. The proof is similary obtained as in the case of rings.

Components and isolated components of a are defined in the obvious way[8].

THEOREM 9. *The isolated component $a'=q_{i_1}\cap\cdots\cap q_{i_r}$ of a is uniquely determined by the set $\{rad(q_{i_1}),\cdots,rad(q_{i_r})\}$. In particular the isolated primary component of a is uniquely determined.*

THEOREM 10. *Let p be a maximal (divisor-free) prime element of L. Then $a(\neq e)$ is p-primary if and only if $p^{\rho}\leq a$ for a suitable whole number ρ.*

The proofs of the above two theorems are similarly obtained as in the case of rings.

8) *E. g. cf.* [15].

§ 6. Pairwise coprime decomposition.

In this section we shall assume the following conditions:

1°) e is the (multiplicative) unit element of L.

2°) L is modular as a lattice.

The multiplication is not necessarily associative. An element a is said to be coprime to b if $a \cup b = e$. Let S be any fixed subset of L. An element a is said to be coprime to S, if a is coprime to every $x \in S$. Then it is easily verified that the set of all elements coprime to S forms an m-sublattice of L. In particular, if $a_1, \cdots, a_n$ are coprime to S, then any product of $a_1, \cdots, a_n$ is coprime to S.

Let $A = \{a_1, \cdots, a_\lambda\}$, $B = \{b_1, \cdots, b_\mu\}$ be two finite subsets of L, and suppose that a_i $(i=1, \cdots, \lambda)$ are coprime to B, and that b_k $(k=1, \cdots, \mu)$ are coprime to A. Then it is easily verified that any product of $a_1, \cdots, a_\lambda$ is coprime to any product of $b_1, \cdots, b_\mu$, and also coprime to $b_1 \cap \cdots \cap b_\mu$.

LEMMA 13. *If $a_1, \cdots, a_\lambda$ are pairwise coprime, then an arbitrary product of $a_1, \cdots, a_\lambda$ is equal to $a_1 \cap \cdots \cap a_\lambda$.*

Proof. If $\lambda = 2$, the lemma is easily obtained. The proof is completed by induction on λ, Q.E.D.

If $a_1, \cdots, a_\lambda$ are pairwise coprime, then by Lemma 13 we can omit the parentheses in any product of $a_1, \cdots, a_\lambda$.

LEMMA 14. *Put $a_i^* = a_1 \cdots a_{i-1} a_{i+1} \cdots a_\lambda$ $(i=1, \cdots, \lambda)$ for pairwise coprime elements $a_1, \cdots, a_\lambda$ of L. Then*

(1) $a_1^* \cup \cdots \cup a_\nu^* = a_{\nu+1} \cdots a_\lambda$ *for* $\nu = 1, \cdots, \lambda-1$.

(2) $a_1^* \cup \cdots \cup a_\lambda^* = e$.

(3) $a_1^*, \cdots, a_\lambda^*$ *are independent over* $a_1 \cap \cdots \cap a_\lambda$.

Proof. These are immediate.

In the following $[a, b]$ will denote the closed interval $\{x;\ a \leq x \leq b,\ x \in L\}$. Let a be any fixed element of L, and let $\mathfrak{j}$ be the set of all closed intervals $[u, a]$, $[v, a]$, $\cdots$. We now define

$$[u, a] \cup [v, a] = [u \cup v, a],$$
$$[u, a] \cap [v, a] = [u \cap v, a],$$
$$[u, a] \odot [v, a] = \begin{cases} [uv, a] & \text{if } uv \geq a, \\ [a, a] & \text{if } uv < a. \end{cases}$$

Then $\mathfrak{j}$ forms an m-lattice under the above defined join, meet and multiplication. $\mathfrak{j}$ is evidently isomorphic to $[e, a]$ as a lattice.

If $x_1, \cdots, x_\lambda$ are independent over a, then $[x_1, a] \cup \cdots \cup [x_\lambda, a]$ is called a direct join of $[x_i, a]$; in symbols $[x_1, a] + \cdots + [x_\lambda, a]$. If $[x, a] = [x_1, a] + [x_2, a]$, $x_1 \neq x$, $x_2 \neq x$, then $[x, a]$ is said to be directly decomposable. Otherwise $[x, a]$ is said to be directly indecomposable.

LEMMA 15. *If $[e, a]$ is a direct join of a finite number of direct indecomposable intervals, then such a decomposition is uniquely determined apart from the commutativity of join.*

Proof. Suppose that $[e, a]=[a_1, a]+\cdots+[a_n, a]=[b_1, a]+\cdots+[b_m, a]$ are two direct decompositions of $[e, a]$ such that every $[a_i, a]$ and $[b_k, a]$ are directly indecomposable. Then $[a_i, a]=[a_ie, a]=[a_i(b_1\cup\cdots\cup b_m), a]=[a_ib_1\cup\cdots\cup a_ib_m, a]=[a_ib_1, a]\cup\cdots\cup[a_ib_m, a]$. Since $[a_ib_k, a]\subseteq[b_k, a]$ $(k=1,\cdots,m)$, we have $[a_i, a]=[a_ib_j, a]$ and $a_ib_k\leq a$ for $k\neq j$. Next we have $[b_i, a]=[eb_j, a]=[(a_1\cup\cdots\cup a_n)b_j, a]=[a_1b_j\cup\cdots\cup a_nb_j, a]=[a_1b_j, a]\cup\cdots\cup[a_nb_j, a]$. Since $[a_lb_j, a]\subseteq[a_l, a]$ $(l=1,\cdots,n)$, $[b_j, a]$ is equal to $[a_ib_j, a]$. Hence $[b_j, a]\subseteq[a_l, a]$, i.e., each $[a_i, a]$ is equal to some $[b_j, a]$. This completes the proof.

THEOREM 11. *Suppose that a.c.c. holds for L. Then every element a $(\neq 0)$ is uniquely decomposed as a meet (product) of a finite number of pairwise coprime elements, none of which has such decomposition.*

Proof. It is easy to see that any element $a(\neq 0)$ is decomposed as a meet of a finite number of pairwise coprime elements, none of which has such decomposition:

$$a=a_1\cap\cdots\cap a_n,\quad a_i\cup a_j=e\ (i\neq j).$$

We now prove the uniqueness of this decomposition. Put $a_i^*=a_1\cdots a_{i-1}a_{i+1}\cdots a_n$ $(i=1,\cdots,n)$. Then

$$(*)\qquad [e, a]=[a_1^*, a]+\cdots+[a_n^*, a].$$

The lattice-quotient a_i^*/a is clearly transposable to e/a_i. This implies a lattice-isomorphism between $[a_i^*, a]$ and $[e, a_i]$. If $[e, a_i]$ is directly decomposable: $[e, a_i]=[b, a_i]+[c, a_i]$, $b\neq e$, $c\neq e$, then $e=b\cup c$, $a_i=b\cap c$. This is a contradiction. Hence $[e, a_i]$ is directly indecomposable, and hence so is $[a_i^*, a]$. By Lemma 15 the decomposition $(*)$ is uniquely determined, and so are $a_i=a_1^*\cup\cdots\cup a_{i-1}^*\cup a_{i+1}^*\cup\cdots\cup a_n^*$. This completes the proof.

REMARK 3. Suppose that a.c.c. holds for L. Let $a=q_1\cap\cdots\cap q_r$ be a short representation of $a(\neq 0)$. Two components q_i and q_k of a are called equivalent, if we can chose primary components $q_{i(0)}, q_{i(1)},\cdots,q_{i(t)}$ of a such that $q_i=q_{i(0)},\cdots,q_{i(t)}=q_k$ and $q_{i(j)}\cup q_{i(j+1)}=e$ for $j=0,\cdots,t-1$. Let a_i be the meet of the primary components of a which are equivalent to q_i $(i=1,\cdots,n)$. Then $a_1\cap\cdots\cap a_n=a$ is the pairwise decomposition of a.

§7. r-Ideals in multiplicative systems.

A set M is called a multiplicative system (m-system), if M has a binary operation $(\cdot)$. If the operation is commutative, M is called a commutative m-system; if the operation is associative, M is called a semigroup.

Let M be a commutative m-system with a zero element. A subset A of M is called an s-ideal if $MA=\{xa;\ x\in M,\ a\in A\}\subseteq A$. The set $\mathfrak{L}$ of all s-ideals of any commutative m-system forms a cm-lattice under set-inclusion relation and a multiplication $AB=\{ab;\ a\in A,\ b\in B\}$. It is evident that $\mathfrak{L}$ has the properties C_1), C_2) and C_3) in §1. By (2) in §1, $\mathfrak{L}$ forms a residuated lattice.

Let $A\to\bar{A}$ be a mapping from $\mathfrak{L}$ into itself with the following conditions:

1) $A\subseteq\bar{A}$,

2) $\bar{\bar{A}}=\bar{A}$,

3) $A\subseteq B$ implies $\bar{A}\subseteq\bar{B}$,

4) $\bar{A}\bar{B}\subseteq\overline{AB}$,

5) $\overline{(a)}=(a)$[9],

6) $\overline{(a)A}=(a)\bar{A}$.

An s-ideal A is called an r-ideal[10] or a closed ideal (c-ideal) if $\bar{A}=A$. A set $\mathfrak{L}_r$ of all r-ideals in $\mathfrak{L}$ forms a cm-lattice under set-inclusion relation and a multiplication $A\cdot B=\overline{AB}$. Moreover $\mathfrak{L}_r$ forms a residuated lattice. $\mathfrak{L}_r$ is called an r-ideal-system[11] of M.

For each whole number ρ, ρ-th power of an r-ideal A is defined inductively by $A^{(\rho)}=A^{(\rho-1)}\cdot A^{(\rho-1)}$, $A^{(1)}=A$. If M forms a commutative semigroup, then $A^{(\rho)}=A^{2^{\rho-1}}$.

Throughout this section an "ideal" will denote an r-ideal of M. In the following we shall impose a.c.c. upon ideals of M.

Let A be any fixed ideal of M. An ideal generated by the set-union of all ideals X satisfying $X^{(\rho)}\subseteq A$ is called a radical of A; symbol: rad(A). If rad$(A)=A$, then A is called a radical ideal of M. By a.c.c. it is easy to see that $(\mathrm{rad}(A))^{(\lambda)}$ is contained in A for a suitable whole number λ. An ideal is called prime, if for ideals A and B, $A\cdot B\subseteq P$ implies $A\subseteq P$ or $B\subseteq P$. Then, by the theorem in [10], we obtain

THEOREM 12. *Every radical ideal has a unique minimal meet-decomposition into prime ideals.*

An ideal Q is called primary if $A\cdot B\subseteq Q$ and $A\not\subseteq Q$ imply $B^{(\sigma)}\subseteq Q$ for a suitable whole number σ. The radical of a primary ideal of M is not always prime. In fact we can find such an example. But, for convenience, we shall say that Q is rad(Q)-primary or that Q belongs to rad(Q). By Theorems 1 and 2 we have the following:

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9) (a) will denote a principal ideal generated by a, i.e., (a) is the set of all products of a finite number of elements containing a.

10) Cf. [9; p. 536], [7; p. 118].

11) Evidently $\mathfrak{L}$ is an r-ideal-system with respect to the discrete closure. The set of all ideals of a commutative ringoid R [3; p. 203] forms an r-ideal-system of R, if we take the module-generation as an r-operation. Hence the ideals of R is a residuated commutative cm-lattice.

THEOREM 13 *Any meet-irreducible ideal of a commutative m-system is a primary ideal.*

THEOREM 14. *Any ideal A of a commutative m-system is represented as an intersection of a finite number of primary ideals, and the radical of A is equal to the intersection of all the radicals of the primary components of A.*

Any ideal A of M is represented as an intersection of a finite number of primary ideals Q_i with the following conditions: (1) $A=Q_1\wedge\cdots\wedge Q_m$ is irredundant and (2) $\mathrm{rad}(Q_1),\cdots,\mathrm{rad}(Q_m)$ are all different. Such a representation is called a short (or normal) representation of A. By Theorem 3 we can establish

THEOREM 15. *In all short representations of a given ideal of a commutative m-system, the numbers of primary ideals are the same, and the sets of radicals belonging to the same primary ideals are also the same.*

Applying Theorem 4, we have

THEOREM 16. *The set $\mathfrak{A}$ of all ideals A of a commutative m-system which satisfies $A\cdot A=A$ forms a distributive lattice. Any ideal in $\mathfrak{A}$ is uniquely represented as an intersection of a finite number of ideals in $\mathfrak{A}$, none of which is represented as an intersection of ideals in $\mathfrak{A}$.*

Let S be a commutative semigroup with a zero element, and let $\mathfrak{L}_r$ be an r-ideal-system of S. We shall assume that a.c.c. holds for $\mathfrak{L}_r$. Then the results mentioned above are of course valid for $\mathfrak{Z}_r$. By Theorems 7 and 8 we obtain

THEOREM 17. *Let $A=Q_1\wedge\cdots\wedge Q_n$ be a short representation of any ideal A in $\mathfrak{L}_r$. Then any prime ideal in $\mathfrak{L}_r$ which contains A contains some $rad(Q_i)$. In particular, any minimal prime r-ideal of A coincides with some $rad(Q_i)$. And $rad(A)$ is equal to the intersection of all the minimal prime ideals in $\mathfrak{L}_r$ of A.*

THEOREM 18. *Let $A=Q_1\wedge\cdots\wedge Q_m$ and $B=Q'_1\wedge\cdots\wedge Q'_n$ be short representations of ideals A and B in $\mathfrak{L}_r$. Then the following conditions are equivalent:*

1) $A:B=A$.

2) $rad(Q_i)\not\supseteq B$ *for* $i=1,\cdots,m$.

3) $rad(Q_i)\not\supseteq rad(Q'_k)$ *for* $i=1,\cdots,m$; $k=1,\cdots,n$.

Components and isolated components of an ideal in $\mathfrak{L}_r$ are defined in the obvious way. The isolated component $A'=Q_{i_1}\wedge\cdots\wedge Q_{i_r}$ of an ideal A in $\mathfrak{L}_r$ is uniquely determined by the set Q_A, where $Q_A=\{x_{i_j}\,;\,x_{i_j}^{\rho}\in Q_{i_j},\ j=1,\cdots,r\}$. In particular isolated primary component of an ideal in $\mathfrak{L}_r$ is uniquely deterimined.

Let P be a maximal prime ideal in $\mathfrak{L}_r$. Then by Theorem 10 an ideal A $(\neq S)$ is P-primary if and only if $P^{\rho}\subseteq A$ for a suitable number ρ.

We shall now consider pairwise decomposition of ideals (r-ideals) in a commutative m-system.

Let A and B be any two ideals of a commutative m-system M. If, for any element a of $A\vee B$[12], there exists an element b (depending on a) such that $A\vee(a)=A\vee(b)$, $b\in B$, then the lattice of all r-ideals of M forms a modular lattice[13].

We assume now that

(1) The ideal M is the multiplicative unit element of the m-lattice of all r-ideals of M.

(2) The lattice of all r-ideals of M is modular.

The concepts of coprimeness, pairwise coprimeness, independency of r-ideals etc. are defined in the obvious way. Then every r-ideal of M is uniquely represented as an intersection of a finite number of pairwise coprime r-ideals, and none of which has such reprsentations except the trivial case. If the multiplication of r-ideals is associative, in particular if M forms a semigroup, then the pairwise coprime representation of an r-ideal A can be obtained from the short representation of A, which is similar to the case of rings.

§8. Decomposition of normal subgroups of a group.

Let G be a group. The set $L_G=\{A, B, P, \cdots\}$ of all normal subgroups of G forms a residuated lattice under the set-inclusion relation, and the commutator-product, where the commutator-product $A\circ B$ means the subgroup generated by all commutators $a^{-1}b^{-1}ab$ $(a\in A,\ b\in B)$. The multiplication of L_G is of course commutative, but not necessarily associative. A normal subgroup P is called prime, if $A\circ B\subseteq P$ implies $A\subseteq P$ or $B\subseteq P$. A normal subgroup Q is called primary, when $A\circ B\subseteq Q$ and $A\not\subseteq Q$ imply that, for a suitable whole number σ, σ-th derived group of B is contained in Q. It is not true in general that $\mathrm{rad}(Q)$ is prime for a primary normal subgroup Q.

Under the assumption of a.c.c. for normal subgroups of G, the results obtained in §3 are applicable to L_G in the obvious way. In particular we obtain the following

THEOREM 19. *Any normal subgroup A of a group with ascending chain condition for normal subgroups is an intersection of a finite number of primary normal subgroups Q_i with* (1) *$A=Q_1\wedge\cdots\wedge Q_m$ is irredundant and* (2) *$rad(Q_1),\cdots,rad(Q_m)$ are all different. If $A=Q_1\wedge\cdots\wedge Q_m=Q_1'\wedge\cdots\wedge Q_n'$ are such two representations of A, then $m=n$, and $rad(Q_i)$ and $rad(Q_k')$ are equal in pairs.*

Let G be a group equal to its derived group. If a normal subgroup A of G is equal to its derived group and if G/A has a finite principal series, then by Theorem 5 A is represented as an intersection of a finite number of irreducible normal subgroups $A_1,\cdots,A_n$, each of which is equal to its derived group; and A is equal to an arbitrary commutator-product of $A_1,\cdots,A_n$. If a normal subgroup

12) $A\vee B$ will denote an r-ideal generated by A and B.

13) Cf. [1; Lemma 7.3].

A is equal to its derived group, and if G/A has a finite principal series, then by Theorem 6 A is an intersection of a finite number of pairwise complementary subgroups, each of which is equal to its derived group and none of which is represented as an intersection of pairwise complementary subgroups (E. Schenkman)[14].

§ 9. NRP-elements.

Let H be a subset of an upper complete lattice V, and let N be an arbitrary subset of H. If $x \leq \sup[N]$ $(x \in H)$ implies the existence of a finite number of elements $x_1, \cdots, x_n$ of N satisfying $x_1 \cup \cdots \cup x_n \geq x$, then H is called an accessible subset[15] of V.

In this and next two sections we shall use K to denote a commutative or noncommutative cl-semigroup[16] with the following conditions:

(1°) K has the greatest element e.

(2°) K has the zero element 0.

(3°) $ab \leq a$ and $ab \leq b$ for any two elements $a, b \in K$.

We now assume throughout this and the next sections that there exists a subset $\sum (\neq K)$ of K which satisfies the following two conditions:

P_1) Any element a of K is represented as a join of a finite or an infinite number of elements x_λ $(\lambda \in \Lambda)$ of $\sum$, i.e., $a = \cup_{\lambda\in\Lambda} x_\lambda$.

P_2) $\sum$ is an accessible subset of K.

The elements in K will be denoted by $a, b, c, p, q, \cdots$, and the elements in $\sum$ by $x, y, z, u, \cdots$, with or without suffices.

Let a and b be any two elements of K such that $a < b$. Then we can take an element x of $\sum$ which satisfies $x \not\leq a$ and $x \leq b$. For, let $b = \cup_{\lambda\in\Lambda} x_\lambda$, $x_\lambda \in \sum$. Then evidently there exists $x = x_\lambda$ satisfying $x \not\leq a$.

If the ascending chain condition (a. c. c.) holds for the elements of K, then it is evident that any element of K is represented as a join of a finite number of elements in $\sum$.

Definition 3. Let a and b be any two elements of K. The supremum of the elements $\{f\}$ satisfying $feb \leq a$ is called a right residual of a by b. Symbol: $(a:b)_r$. Symmetrically for a left residual $(a:b)_l$ of a by b.

If the greatest element e is a (multiplicative) unit element of K, then the residuals of a by b defined above coincide with those of Birkhoff [3].

Some of the elementary properties of the residuals are listed here.

(1) $a \leq (a:b)_r$, and symmetrically.

(2) $(\cap_\lambda a_\lambda : b)_r = \cap_\lambda (a_\lambda : b)_r$, and symmetrically.

(3) $(a : \cup_\lambda b_\lambda)_r = \cap_\lambda (a : b_\lambda)_r$, and symmetrically.

14) Cf. [14; p. 380].
15) Cf. § 12.
16) Cf. [3; p. 201].

(4) $(ab:b)_r \geq a$, and symmetrically.

(5) $(a:b)_r \geq c$, $(a:c)_l \geq b$ and $ceb \leq a$ are equivalent to one another.

(6) $((a:b)_r:c)_l = ((a:c)_l:b)_r$.

(7) $(a:bec)_r = ((a:c)_r:b)_r$, $(a:bc)_r \leq ((a:c)_r:b)_r$, and symmetrically.

In the following there is complete parallelism between the theory of a right-side and that of a left-side. We shall therefore state the results for the right-side only.

LEMMA 16. *$(a:b)_r$ is equal to the supremum of the elements $\{x\}$ satisfying $xeb \leq a$, $x \in \sum$.*

Proof. Let c be an arbitrary element such that $ceb \leq a$, and let $c = \bigcup_{\lambda\in\Lambda} x_\lambda$, $x_\lambda \in \sum$. Then since $a \geq ceb = \bigcup_{\lambda\in\Lambda}(x_\lambda eb)$, we have $a \geq x_\lambda eb$ for every $\lambda \in \Lambda$. Hence $c \leq t$, where t is the supremum of $\{x\}$ satisfying $xeb \leq a$, $x \in \sum$. In particular $(a:b)_r \leq t$. The converse inclusion is evident.

DEFINITION 4. An element b is said to be relatively (right) prime to a, when $(a:b)_r = a$. Otherwise b is said to be not relatively (right) prime to a, or shortly b is called an NRP-element of a. An element c is said to be relatively (right) weak-prime to a, when $(a:x)_r = a$ for every $x \leq c$, $x \in \sum$. Otherwise c is said to not relatively (right) weak-prime to a, or shortly c is called a weak NRP-element of a.

It is easily verified that an NRP-element of a is always a weak NRP-element of a. But the converse is not true in general. Any weak NRP-element contained in $\sum$ of a is evidently an NRP-element of a. In order that an element b is an NRP-element of a, it is necessary and sufficient that there exists an element $x(\in \sum)$ such that $xeb \leq a$ and $x \not\leq a$.

An element p of K is called prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. Then it is easily verified that in order that $p\ (\neq e)$ is prime, it is necessary and sufficient that $xey \leq p(x, y \in \sum)$ implies $x \leq p$ or $y \leq p$.

DEFINITION 5. If there exists an element p which is maximal in the set of all elements each of which is prime and a weak NRP-element of a, then p is called a (right) maximal prime element of a.

DEFINITION 6. If there exists an element b which is maximal in the set of all the weak NRP-elements of a, then b is called a (right) W-maximal element of a.

LEMMA 17. *Let p be a W-maximal element of a. If p is an NRP-element of a, then it is a maximal prime element of a.*

Proof. It is sufficient to show that p is a prime element. Suppose that $xey \leq p$ and $y \not\leq p$. Since $(a:p)_r > a$, we can take an element z satisfying $z \in \sum$, $z \not\leq a$ and $z \leq (a:p)_r$, i.e., $zep \leq a$.

Now since $y \not\leq p$ and since p is maximal among the weak divisors of a, we can find an element $y_0 \in \Sigma$ such that $y_0 \leq p \cup y$ and $(a:y_0)_r = a$. Then $zex \cdot ey_0 \leq zexe(p \cup y) = zexep \cup ze \cdot xey \leq zep \cup zep \leq a$. Hence $zex \leq (a:y_0)_r = a$. Let x' be an arbitrary element such that $x' \leq x \cup p$ and $x' \in \Sigma$. Then since $ze(x \cup p) = zex \cup zep \leq a$ and $z \not\leq a$, we have $(a:x')_r \geq (a:x \cup p)_r > a$. Hence $x \cup p$ is a weak NRP-element of a. We have therefore $p \cup x = p$, $x \leq p$, as desired.

An element a of K is called strongly meet-irreducible if a is not represented as a meet of a finite or an infinite number of elements containing a strictly. An element a of K is called meet-reducible, if a is not represented as a meet of a finite number of elements containing a strictly.

LEMMA 18. *If a is strongly meet-irreducible, then any weak NRP-element of a is an NRP-element of a.*

Proof. Let d be any weak NRP-element of a, and let u be an arbitrary element of Σ which is contained in d. Then evidently $(a:u)_r > a$ and $\bigcap_{u \leq d}(a:u)_r \geq a$. Hence $\bigcap_{u \leq d}(a:u)_r > a$. Take an element x in Σ such that $x \not\leq a$ and $x \leq \bigcap_{u \leq d}(a:u)_r$. Then of course $xeu \leq a$ for every $u \leq d$. Hence we have $xed = xe(\bigcup_{u \leq d} u) = \bigcup_{u \leq d}(xeu) \leq a$, and $a < (a:d)_r$, completing the proof.

§10. Primal decomposition.

DEFINITION 7. Let $\mathfrak{D}_a$ be the set[17] of all (right) NRP-elements in Σ of a. If the $\sup[\mathfrak{D}_a]$ is a weak NRP-element of a, then a is called a (right) primal element of K. In this case $\sup[\mathfrak{D}_a]$ is called a (right) adjoint element of a. Symbol: $\mathrm{adj}(a)$.

LEMMA 19. *If an arbitrary join of a finite number of elements in $\mathfrak{D}_a$ is an NRP-element of a, then a is a primal element.*

Proof. Take an arbitrary element x such that $x \in \Sigma$ and $x \leq \sup[\mathfrak{D}_a]$. Then there exist $x_1, \cdots, x_n$ such that $x_i \in \mathfrak{D}_a$ and $x \leq \bigcup_{i=1}^n x_i$. Hence $(a:x)_r \geq (a:\bigcup_{i=1}^n x_i)_r > a$.

LEMMA 20. *A meet-irreducible element is a primal element. In particular a strongly meet-irreducible element is primal.*

Proof. Let a be an irreducible element of K, let x be an arbitrary element such that $x \leq \sup[\mathfrak{D}_a]$, $x \in \Sigma$ and let $x_1, \cdots, x_n$ be any finite number of elements such that $x \leq x_1 \cup \cdots \cup x_n$, $x_i \in \mathfrak{D}_a$, i.e., $(a:x_i)_r > a$ $(i=1, \cdots, n)$. If $n > 1$, then since a is meet-irreducible we have $a < \bigcap_{i=1}^n (a:x_i)_r = (a:\bigcup_{i=1}^n x_i)_r \leq (a:x)_r$. If $n=1$, then $a < (a:x_1)_r \leq (a:x)_r$.

LEMMA 21. *If a is strongly meet-irreducible, then $\mathrm{adj}(a)$ is a maximal prime element of a.*

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17) It may happen that $\mathfrak{D}_a$ is vacuous. The join and the meet of a finite number of elements in $\mathfrak{D}_a$ are not necessarily contained in $\mathfrak{D}_a$.

Proof. By Lemma 20 a is a primal element. By Lemma 18 adj(a) is an NRP-element of a. It is easy to see that adj(a) is a W-maximal element of a. Hence by Lemma 17 adj(a) is a maximal prime element of a.

THEOREM 20. *Every element of K is represented as a meet of a finite or an infinite number of strongly meet-irreducible (right) primal elements.*

Proof. Let a be an arbitrary element of K. If $a=e$, then since no NRP-element of e exists, e is a primal element of K. If $a\neq e$, then we can take an element x_0 such that $x_0\not\leq a$, $x_0\in\Sigma$.

First we shall show that there exists a primal element q such that $x_0\not\leq q$ and $a\leq q$. Let $g_1\leq g_2\leq\cdots$ be any ascending chain with $x_0\not\leq g_i$ $(i=1,2,\cdots)$. Then $x_0\not\leq\sup_i[g_i]$. Because, if we suppose that $x_0\leq\sup_i[g_i]$, then $x_0\leq\cup_{j=1}^n x_j$ for suitable elements x_j contained in the set $\{x_{\lambda_i}^i;\, g_i=\cup_{\lambda_i}x_{\lambda_i}^i,\, x_{\lambda_i}^i\in\Sigma\}$. Take g_{i_j} such that $g_{i_j}\geq x_j$ $(j=1,\cdots,n;\, i_1\leq\cdots\leq i_n)$. Then $x_0\leq x_1\cup\cdots\cup g_{i_n}=g_{i_n}$. This is a contradiction. Zorn's lemma assures therefore the existence of an element q such that (1) a is contained in q, (2) $x_0\not\leq q$ and (3) $q<c$ implies $x_0\leq c$. It is easily verified that q is strongly meet-irreducible. Hence q is primal with $a\leq q$ and $x_0\not\leq q$.

Next we prove that a is equal to the meet $\cap_\lambda q_\lambda$ of all primal elements q_λ satisfying $a\leq q_\lambda$ and $x\not\leq q_\lambda$ for an element x in Σ with $x\not\leq a$. Suppose that $a<\cap_\lambda q_\lambda$. Then we can find an element y such that $y\in\Sigma$, $y\not\leq a$ and $y\leq\cap_\lambda q_\lambda$. Hence of course $y\leq q_\lambda$ for every λ. This contradicts the existence of a primal element $q_{\lambda'}$ satisfying $q_{\lambda'}\not\geq y$. This completes the proof.

LEMMA 22. *Let $a=q_1\cap\cdots\cap q_n$ be a decomposition into primal elements q_i with prime adjoints $p_i=adj(q_i)$. If b is a weak NRP-element of a, then there exists p_i such that $p_i\geq b$.*

Proof. Suppose that b is a weak NRP-element of a. Then $(a:x)_r>a$ for every x satisfying $x\leq b$, $x\in\Sigma$. Therefore we can take an element y depending on x such that $y\not\leq a$ and $y\leq(a:x)_r$. Hence $yex\leq a\leq q_i$ for all $i=1,\cdots,n$. Since there exists q_j such that $y\not\leq q_j$, we have $q_j<(q_j:x)_r$, i.e., $x\in\mathfrak{D}_{q_j}$. This implies $b=\sup_{x\leq b}[x]\leq\cup_j\sup[\mathfrak{D}_{q_j}]=\cup_j p_j$ and

$$j(b)\subseteq j(p_1)\cup\cdots\cup j(p_n)\,,$$

where $j(b)$, $j(p_i)$ denote the principal (lattice-)ideal generated by b, p_i respectively, and $\cup$ denotes the set-theoretical union. Suppose now that $j(b)\subseteq j(p_1)\cup\cdots\cup j(p_k)$ and $j(b)\not\subseteq j(p_1)\cup\cdots\cup j(p_{\sigma-1})\cup j(p_{\sigma+1})\cup\cdots\cup j(p_k)$ $(1\leq\sigma\leq k)$. Then we can find an element c_i such that $c_i\in j(b)\wedge j(p_i)$ and $c_i\notin j(p_\sigma)$ $(\sigma\neq i)$ for every $i=1,\cdots,k$.

Now we prove that $k=1$, Suppose that $k>1$. Since $c=c_1\cdots c_{k-1}\cup c_k\leq b$, c is contained in some $j(p_\sigma)$, $\sigma\leq k$. If $\sigma\neq k$, then $c_k\leq c\leq p_\sigma$, hence $c_k\in j(p_\sigma)$, a contradiction. Hence $\sigma=k$. This implies $c\cdots c_{k-1}\leq p_k$, and $c_i\leq p_k$ for some $i<k$ which is also a contradiction.

DEFINITION 8. A meet-decomposition $a=a_1\cap\cdots\cap a_n$ is called reduced or maximal, if no a_i can be replaced by a'_i containing a_i properly.

LEMMA 23. *If a.c.c. holds for K, then any element of K has an irredundantly maximal meet-decomposition.*

Proof. This is immediate.

LEMMA 24. *Let $a=q_1\cap\cdots\cap q_n$ be a maximal meet-decomposition into primal elements q_i with adjoints $p_i=adj(q_i)$. If $b\leq p_i$ for some p_i, then b is a weak NRP-element of a. In particular every p_i is a weak NRP-element of a.*

Proof. We can suppose without loss of generality that $b\leq p_1$. Take an arbitrary element z with $z\leq p_1$ $(z\in\sum)$, and moreover take an element y with $y\not\leq q_1$ $(y\in\sum)$, $yez\leq q_1$. Then $q_1^*\equiv q_1\cup y>q_1$. Since $a=q_1\cap\cdots\cap q_n$ is maximal, we can choose an element u in $\sum$ such that $u\not\leq a$ and $u\leq q_1^*\cap q_2\cap\cdots\cap q_n$. Now since $u\leq q_1^*$, we have $uez\leq(q_1\cup y)ez=q_1ez\cup yez\leq q_1\cup q_1=q_1$. On the other hand, $uez\leq u\leq q_2\cap\cdots\cap q_n$. Hence $uez\leq q_1\cap q_2\cap\cdots\cap q_n=a$. p_1 is therefore a weak NRP-element of a. Hence b is of course a weak NRP-element of a. This completes the proof.

THEOREM 21. *If $a=q_1\cap\cdots\cap q_n$ is a maximal meet-decomposition of a into primal elements q_i with prime adjoints $p_i=adj(q_i)$, then the W-maximal elements of a, the maximal prime elements of a and the maximal elements in the set $\{p_1,\cdots,p_n\}$ are the same.*

Proof. Let p be a W-maximal element of a. Then by Lemma 22 p is contained in some p_i. Since by Lemma 24 p_i is a weak NRP-element of a, we have $p=p_i$. Hence p is prime and hence a maximal prime element of a. Let p_j be any maximal element in the set $\{p_1,\cdots,p_n\}$. Then p_j is a W-maximal element of a. For, if not, then there exists a weak NRP-element c of a such that $p_j<c$. Now by Lemma 22 we have $c\leq p_k$ for some p_k. Hence $p_j<p_k$. This is a contradiction. Again by Lemma 22 a maximal prime element of a is maximal in the set $\{p_1,\cdots,p_n\}$. By Lemma 22 it is easily verified that a maximal element in $\{p_1,\cdots,p_n\}$ is a W-maximal element of a.

THEOREM 22. *If $a=q_1\cap\cdots\cap q_n$ is a maximal decomposition of a into primal elements q_i with prime adjoints $p_i=adj(q_i)$, then a is primal if and only if one p_j contains all the others, and $adj(a)=p_j$.*

Proof. Let $p_j\geq p_i$ for all i, then $p_j=p_1\cup\cdots\cup p_n$. Since by Lemma 24 p_j is a weak NRP-element of a, we have $P=\{x;x\leq p_j\}\subseteq\mathfrak{D}_a$, $p_j=\sup_x[P]\leq\sup[\mathfrak{D}_a]$. On the other hand, since by Lemma 22 every element in $\mathfrak{D}_a$ is contained in p_j, we have $\sup[\mathfrak{D}_a]\leq p_j$. Hence $\sup[\mathfrak{D}_a]=p_j$. This implies that a is primal with prime adjoint p_j. Conversely, let a be a primal element. Then by Lemma 22 $\mathrm{adj}(a)\leq p_j$ for some j. Since by Lemma 24 p_i is a weak NRP-element of a, we have $p_i \leq \mathrm{adj}(a)$ for all i, and have $p_j \leq \mathrm{adj}(a) \leq p_j$. This implies that $\mathrm{adj}(a) = p_j$.

DEFINITION 9. An irredundant decomposition $a = q_1 \cap \cdots \cap q_n$ of a into primal elements q_i is called a short (or normal) representation of a, if every $q_k \cap q_l$ $(k \neq l)$ is not primal.

THEOREM 23. *If a has a maximal decomposition by primal elements with prime adjoints, then a has a short representation by primal elements whose adjoints are just the maximal primes of a.*

Proof. Let

$$(*) \qquad a = q_1 \cap \cdots \cap q_n$$

be a maximal decomposition of a into primal elements q_i with prime adjoints $p_i = \mathrm{adj}(q_i)$. Assume that $(*)$ is irredundant. We now suppose without loss of generality that $p_1, \cdots, p_t$ $(t \leq n)$ are the maximal elements in $\{p_1, \cdots, p_n\}$. Let q_1^* be the meet of those q_i such that $p_i \leq p_1$, and q_j^* the meet of those q_i such that $p_i \leq p_j$ and $p_i \not\leq p_k$ if $k < j (j = 2, \cdots, t)$. Since q_j^* satisfies the conditions of Theorem 22, p_j is the prime adjoint of q_j $(j = 1, \cdots, t)$. Let $q_k^* = q_{k_1} \cap \cdots \cap q_{k_s}$ and $q_l^* = q_{l_1} \cap \cdots \cap q_{l_g}$. Then $q_k^* \cap q_l^* = q_{k_1} \cap \cdots \cap q_{k_s} \cap q_{l_1} \cap \cdots \cap q_{l_g}$ is of course a maximal decomposition into primal elements, and whose adjoints are not all contained in any one adjoint. Hence by Theorem 22 $q_k^* \cap q_l^*$ is not primal. $a = q_1^* \cap \cdots \cap q_t^*$ is therefore a short representation of a. By Theorem 21 $p_1, \cdots, p_t$ are the maximal primes of a, Q. E. D.

If K is modular then Hilfssatz II in [13] is valid for K. Hence by Theorem 23 we obtain

THEOREM 24. *Suppose that K is modular. If an element a of K is represented as a meet of a finite number of strongly meet-irreducible elements then a has a short representation into primal elements whose adjoints are the maximal primes of a.*

THEOREM 25. *Let $a = q_1 \cap \cdots \cap q_m = q_1' \cap \cdots \cap q_n'$ be two short representations of a into primal elements q_i and q_k' with prime adjoints $p_i = adj(q_i)$ and $p_k' = adj(q_k')$. Then $m = n$ and the two sets of the p_i's and the p_k''s are the same.*

Proof. No p_i contains another p_j strictly. Hence by Theorem 21 $\{p_1, \cdots, p_m\}$ is equal to the set $\mathfrak{P}_a$ of all maximal primes of a. Similarly $\{p_1', \cdots, p_n'\} = \mathfrak{P}_a$. The theorem is now clear.

THEOREM 26. *Let $a = q_1 \cap \cdots \cap q_n$ be a maximal decomposition of a into primal elements q_i with prime adjoints $p_i = adj(q_i)$, and let p_i be an NRP-element of q_i for $i = 1, \cdots, n$. Then in order that an element b is a weak NRP-element of a, it is necessary and sufficient that b is contained in some p_i. And every weak NRP-element of a is an NRP-element of a.*

Proof. Since p_1 is an NRP-element of q, we can find an element x such that $x \in \sum$, $x \not\leq q_1$ and $x \leq (q_1 : p_1)_r$. Put $q_1^* = q_1 \cup x$. Then since $q_1^* > q_1$ we have $q_1^* \cap q_2$

$\cap \cdots \cap q_n > a$. Take an element y such that $y \in \Sigma$, $y \not\leq a$ and $y \leq q_1^* \cap q_2 \cap \cdots \cap q_n$. Then $yep_1 \leq (q_1^* \cap q_2 \cap \cdots \cap q_n)$. $ep_1 \leq q_1^* ep_1 \cap q_2 \cap \cdots \cap q_n = (q_1 \cup x)ep_1 \cap q_2 \cap \cdots \cap q_n = (q_1 ep_1 \cup xep_1) \cap q_2 \cap \cdots \cap q_n \leq q_1 \cap q_2 \cap \cdots \cap q_n = a$, i.e., $y \leq (a : p_1)_r$. Since $y \not\leq a$, we have $a < (a : p_1)_r$. p_1 is therefore an NRP-element of a. Similarly any p_i is an NRP-element of a. By Lemma 17 the p_i are all prime, and by Lemmas 22 and 24 b is an NRP-element of a if and only if $b \leq p_i$ for some p_i. Now since $(a : b)_r \geq (a : p_i)_r > a$, we complete the proof.

Let $\mathfrak{D}_a^*$ be the set of all weak NRP-elements a^* of a which is represented as a join of a finite number of elements in Σ and let $\mathfrak{F}_a$ be the set of all elements z in Σ such that $z \cup a^*$ is a weak NRP-element of a for all a^* in $\mathfrak{D}_a^*$. Then it is easy to see that $\mathfrak{F}_a$ is contained in $\mathfrak{D}_a$. If a is primal, then $\mathfrak{F}_a = \mathfrak{D}_a$; hence $\mathrm{adj}(a)$ defined above is equal to $\sup [\mathfrak{F}_a]$.

Definition 10. For every element a of K, $\sup [\mathfrak{F}_a]$ is called a (right) adjoint element of a, and denoted by $\mathrm{adj}(a)$.

Theorem 27. *Adj(a) of every element a of K is represented as the meet of all W-maximal elements of a.*

Proof. Let c be the meet of all W-maximal elements of a, let x be any element in Σ such that $x \leq c$, and let a^* be an arbitrary element in $\mathfrak{D}_a^*$. Then it is easily verified that $\sup_i [a_i]$ is a weak NRP-element of a for any ascending chain $a^* \leq a_1 \leq a_2 \leq \cdots$ consisting of weak NRP-elements of a. Hence by Zorn's lemma, we can take a W-maximal element b of a which contains a^*. Then $x \cup a^* \leq b$. Hence $x \cup a^*$ is a weak NRP-element of a. Therefore $x \in \mathfrak{F}_a$, $x \leq \mathrm{adj}(a)$. This implies $c = \bigcup_{x \leq c} x \leq \mathrm{adj}(a)$. Conversely, let $b = \bigcup_{\lambda \in \Lambda} y_\lambda$ be any weak NRP-element of a, and let x be any element satisfying $x \leq \mathrm{adj}(a)$, $x \in \Sigma$. Take an arbitrary element y such that $y \leq x \cup b, y \in \Sigma$. Then we can take a finite number of elements $y_1, \cdots, y_n$ in $\{y_\lambda ; \lambda \in \Lambda\}$ such that $y \leq x \cup y_1 \cup \cdots \cup y_n$. Since it is easily verified that $y_1 \cup \cdots \cup y_n$ is contained in $\mathfrak{D}_a^*$, $x \cup y_1 \cup \cdots \cup y_n$ is a weak NRP-element of a by the definition of $\mathrm{adj}(a)$. Hence $(a : y)_r > a$ and hence $x \cup b$ is a weak NRP-element of a. Therefore $\mathrm{adj}(a)$ is contained in every W-maximal element of a. This completes the proof.

Throughout the rest of this section we shall assume that

P_3) K satisfies the ascending chain condition for elements.

P_4) K is modular as a lattice.

It is easily verified that every element of K is represented as a join of a finite number of elements in Σ.

Theorem 28. *Let a be a meet-irreducible element. If b is a weak NRP-element of a, then b is an NRP-element of a.*

Proof. Let $b = \bigcup_{i=1}^n y_i$, $y_i \in \Sigma$. It is evident that $a \leq \bigcap_{i=1}^n (a : y_i)_r$. If $n = 1$, then since b is a weak NRP-element of a, we have $a<(a:y_1)_r$; if $n>1$, then since a is meet-irreducible, we have $a<\bigcap_{i=1}^n (a:y_i)_r=(a:\bigcup_{i=1}^n y_i)_r=(a:b)_r$, completing the proof.

THEOREM 29. *Every element a of K has a short maximal meet-decomposition into primal elements whose adjoints are the maximal primes of a and are NRP-elements of a.*

Proof. By a.c.c. any element a of K is represented as

$$(*) \qquad a=a_1\cap\cdots\cap a_n,$$

where each a_i are meet-irreducible. Suppose that $(*)$ is irredundantly maximal. Then by Lemma 20 each a_i is primal. By Theorem 28 $p_i=\mathrm{adj}(a_i)$ is an NRP-element of a, and by Lemma 17 p_i is a prime element. Hence by Theorem 26 p_i is an NRP-element of a. Suppose that $p_1,\cdots,p_t$ are the maximal elements in the set $\{p_1,\cdots,p_n\}$. Let q_j be the meet of a_i such that $p_i\leq p_j$ and $p_k\not\leq p_j$ for $k<j$, $j=1,\cdots,t$. Then by Lemma 20 q_j is primal. Since it is verified that $a=q_1\cap\cdots\cap q_t$ is maximal[18], by Theorems 21 and 22 we complete the proof.

Using Theorems 21 and 29, we have the following two theorems:

THEOREM 30. *Any element a of K has a finite number of maximal primes, which are just the maximal weak NRP-elements of a.*

THEOREM 31. *The adjoint of any primal element is a prime element.*

THEOREM 32. *Let a be any element of K satisfying a.c.c. Then any weak NRP-element of a is an NRP-element of a.*

Proof. Let b be any weak NRP-element of a. Then b is contained in an element p such that p is a W-maximal element of a. Since p is a maximal prime of a; p is an NRP-element of a. Hence there exists x such that $x\in\sum$, $x\not\leq a$, and $xep\leq a$, hence $xeb\leq xep\leq a$, i.e., b is an NRP-element of a.

THEOREM 33. *The adjoint of every element a of K is represented as the meet of the maximal prime elements of a.*

Proof. Every element of K has a maximal (reduced) meet-decomposition into primal elements. Hence the result follows at once from Theorem 27.

§11. Isolated p-components.

In the present section we shall impose upon K the conditions $P_1)$, $P_2)$, and

$C)$ K is lower complete.

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18) $a=q_1\cap a_{i_1}\cap\cdots\cap a_{i_r}$, $p_{i_l}\not\leq p_1$ $(l=1,\cdots,r)$ is maximal. In any modular lattice with a.c.c. we can prove the following, which is quite similar to the case of rings. (See [13; Hilfssats IV, pp. 36–37]). If $a=c_{11}\cap\cdots\cap c_{1\mu_1}\cap\cdots\cap c_{\sigma 1}\cap\cdots\cap c_{\sigma\mu_\sigma}$ is maximal (reduced), then $a=b_1\cap\cdots\cap b_\sigma$ is also maximal, where $b_i=c_{i1}\cap\cdots\cap c_{i\mu_i}$.

DEFINITION 11. Let p be a prime element containing a. The *(right upper) isolated p-component* $a(p)$ of a is defined as the meet of all elements b such that (1) $a \leq b$ and (2) $x \not\leq p$ $(x \in \sum)$ implies $(a:x)_r = b$. In particular $a(e) = a$.

LEMMA 25. *If a is primal with prime adjoint p, then $a(p) = a$.*

Proof. Since $p = \mathrm{adj}(a)$ is a weak NRP-element of a, the lemma is immediate.

LEMMA 26. *If $a \leq a'$, and $a' \leq p$, then $a(p) \leq a'(p)$.*

Proof. Let B_p be the set of all b such that $x \not\leq p$ $(x \in \sum)$ implies $(b:x)_r = b$, and let $B_p(a)$ the set of all b which satisfy $b \in B_p$ and $b \geq a$. Then it is evident that $a \leq a'$ implies $B_p(a) \supseteq B_p(a')$. Hence $a(p) = \inf[B_p(a)] \leq \inf[B_p(a')] = a'(p)$.

THEOREM 34. *Let $a = \cap_{\lambda \in \Lambda} a_\lambda$ be a decomposition into strongly meet-irreducible primal elements a_λ, and let $p_\lambda = adj(a_\lambda)$. Then*

$$a = \cap_{\lambda \in \Lambda} a(p_\lambda) .$$

Proof. By Lemma 25 $a_\lambda = a_\lambda(p_\lambda)$. Since $a \leq a_\lambda$ and $a_\lambda \leq p_\lambda$, we obtain $a = \cap_\lambda a_\lambda = \cap_\lambda a_\lambda(p_\lambda) \geq \cap_\lambda a(p_\lambda) \geq a$, $a = \cap_\lambda a(p_\lambda)$.

THEOREM 35. *Suppose that K satisfies P_3) and P_4). Then every element a of K is represented as*

$$a = a(p_1) \cap \cdots \cap a(p_n) ,$$

where $p_1, \cdots, p_n$ are the maximal prime elements of a.

Proof. Let $a = q_1 \cap \cdots \cap q_n$ be a decomposition of a into primal elements q_i with $p_i = \mathrm{adj}(q_i)$. Then $a \leq a(p_i) \leq q_i(p_i) = q_i$. This implies $a \leq \cap_{i=1}^n a(p_i) \leq \cap_{i=1}^n q_i = a$, $a = \cap_{i=1}^n a(p_i)$.

LEMMA 27. *If a has a short maximal decomposition into primal elements $q_1, \cdots, q_n$ such that $p_i = adj(q_i)$ is an NRP-element of q_i, then p_i is an NRP-element of $a(p_i)$ for $i = 1, \cdots, n$.*

Proof. By Lemma 17 every p_i is a prime element, and by Theorem 26 every p_i is an NRP-element of a. Hence there exists $x_i \not\leq a$ and $x_i e p \leq a$, and hence $x_i e p_i \leq a(p_k)$ for every k. It is evident that there exists q_l such that $q_l \not\geq x_i$. Hence we have that $x_i e p_i \leq q_l$. This implies $p_i \leq p_l$, $p_i = p_l$ and implies $i = l$. Since $q_i = q_i(p_i) \geq a(p_i)$, we obtain that $a(p_i) \not\geq x_i$, that is, p_i is an NRP-element of $a(p_i)$.

THEOREM 36. *Suppose that any weak NRP-element of an arbitrary element a is an NRP-element of a. If a has a short maximal meet-decomposition into primal elements q_i with $p_i = adj(q_i)$ $(i = 1, \cdots, n)$, then, in order that a prime element p is a maximal prime of a, it is necessary and sufficient that p is a maximal element in $\{p_1, \cdots, p_n\}$.*

Proof. By Lemma 27 p_i is an NRP-element of $a(p_i)$, and by Theorem 21

p_i are the W-maximal NRP-elements of a, hence the maximal primes of a. It is easily verified that if a prime element p containing a is a weak NRP-element of a and also a weak NRP-element of $a(p)$, then p is equal to some p_i. The converse is easy to see.

THEOREM 37. *Suppose that P_3) and P_4) hold for K. Then the maximal prime elements p of a are the maximal NRP-elements of both a and $a(p)$.*

Proof. This is immediate by Theorems 29, 34 and 36.

§12. r-Ideals in semigroups.

Let S be a (noncommutative) semigroup with the zero. A subset A of S is called a left (right) s-ideal of S, if $SA=\{xa\,;\, x\in S,\, a\in A\}\subseteq A$ $(AS=\{ay\,;\, a\in A, y\in S\}\subseteq A)$. A left and a right s-ideal is called a two-sided s-ideal or simply an s-ideal of S. The set S of all s-ideals forms a residuated lattice under residuation: $(A:B)_r=\{(x)$[19]$\,;\,(x)SB\subseteq A\}$, $(A:B)_l=\{(y)\,;\,BS(y)\subseteq A\}$.

We consider a mapping $A\rightarrow\bar{A}$ from S into itself with the following conditions:

1) $A\subseteq\bar{A}$,
2) $\bar{\bar{A}}=\bar{A}$,
3) $A\subseteq B$ implies $\bar{A}\subseteq\bar{B}$,
4) $\bar{A}\bar{B}\subseteq\overline{AB}$,
5) $\overline{(a)}=(a)$,
6) $\overline{(a)A}=(a)\bar{A}$, $\overline{A(a)}=\bar{A}(a)$.

An s-ideal A is called an r-ideal or a closed ideal of S if $\bar{A}=A$.

Let $\mathfrak{o}$ be a (commutative or noncommutative) ring with or without a unit element, and let $\mathfrak{M}$ be a left $\mathfrak{o}$-module. Then the set $\mathfrak{H}$ of all left cyclic $\mathfrak{o}$-submodules of $\mathfrak{M}$ forms an accessible subset of the lattice $\mathfrak{B}$ consisting of all left $\mathfrak{o}$-submodules of $\mathfrak{M}$. Because, let N be an arbitrary subset of $\mathfrak{H}$, and let $(u)\subseteq\sup[N]=$(left $\mathfrak{o}$-module generated by "the subset N of $\mathfrak{M}$"), where (u) is a cyclic left $\mathfrak{o}$-submodule of $\mathfrak{M}$, i.e., $(u)=\mathfrak{o}u+Ru$(R: the ring of integers). Then u is represented as $u=\alpha_1u_1+\cdots+\alpha_nu_n+m_1v_1+\cdots+m_tv_t$ where $\alpha_i\in\mathfrak{o}$, $m_k\in R$ and u_i, v_k are the elements in "the subset N of $\mathfrak{M}$". Hence we have that $(u)=\mathfrak{o}u+Ru\subseteq\mathfrak{o}\alpha_1u_1+\cdots+\mathfrak{o}\alpha_nu_n+\mathfrak{o}m_1v_1+\cdots+\mathfrak{o}m_tv_t\subseteq\mathfrak{o}u_1+\cdots+\mathfrak{o}u_n+\mathfrak{o}v_1+\cdots+\mathfrak{o}v_t\subseteq\mathfrak{o}u_1+Ru_1+\cdots+\mathfrak{o}u_n+Ru_n+\mathfrak{o}v_1+Rv_1+\cdots+\mathfrak{o}v_t+Rv_t=(u_1)+\cdots+(u_n)+(v_1)+\cdots+(v_t)$, where (u_i), $(v_k)\in N$.

In particular the set of all left principal ideals of any ring (with or without the unit element) forms an accessible subset of the lattice of all left ideals of the ring. Similar assertions are true for right ideals, and also for two-sided ideals.

If we assume that the set of all principal (two-sided) r-ideals of S forms an

19) (x) will denote the principal s-ideal generated by x, i.e., (x) is the set-union of SxS, Sx, xS and x.

accessible subset of the lattice of all (two-sided) r-ideals of S, then we can define an NRP-r-ideal, a weak NRP-r-ideal, a W-maximal r-ideal and a maximal prime r-ideal of an r-ideal of S, and also can define a primal r-ideal and its adjoint r-ideal. Then the results obtained in §§ 9, 10 and 11 are applied to the r-ideals of S in the obvious way. Moreover applying these results to an associative ring with or without the unit element, we shall obtain a decomposition theorem for two-sided ideals of the ring, which is somewhat different from the Barnes' primal decomposition theorem [2], but it is a generalization of the Curtis' one [4].

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